About quantum fluctuations and holographic principle in $(4+n)$-dimensional spacetime

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Abstract. – In the article we present explicit expressions for quantum fluctuations of spacetime in the case of $(4+n)$-dimensional spacetimes, and consider their holographic properties and some implications for clocks, black holes and computation. We also consider quantum fluctuations and their holographic properties in ADD model and estimate the typical size and mass of the clock to be used in precise measurements of spacetime fluctuations. Numerical estimations of phase incoherence of light from extra-galactic sources in ADD model are also presented.

Introduction. – In this article we investigate quantum fluctuations in the spacetimes with extra spatial dimensions. In the first section, following Ng Y.J. and van Dam H., we introduce explicit expressions for spacetime fluctuations in $(4+n)$-dimensional spacetime, present the implications for clocks and black holes and examine the holographic properties for introduced fluctuations. Although our derivation of the fluctuation expressions differs from that proposed in [1], the results agree with each other. In the second section we investigate spacetime fluctuations in the Arkani-Hamed-Dimopoulos-Dvali (ADD) model [2]. We show that in case of two extra dimensions fluctuations on any distance in the observable universe are small as compared with the size of compact dimensions, but in the case of 11-dimensional spacetime fluctuations on the size of the observable universe become comparable with the size of compact extra dimensions. We also estimate the parameters of the clocks that can be used in precise distance measurements and show that for $n = 2$ clock's size is much less as compared with the size of extra dimensions. In this section we also investigate holographic properties for the fluctuations, and contrary to the conclusions of [3], explicitly show that holography is not destroyed in ADD model. And finally we present numerical estimations of the phase incoherence of light from extra-galactic sources in ADD model.

Before proceeding to next section let us recall some known facts for the black holes in the $(4+n)$-dimensional spacetime. The Schwarzschild solution in $(4+n)$ dimensions has the

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form [4] (we set $c = \hbar = 1$)

$$ds^2 = -\phi(r)\, dt^2 + \phi(r)^{-1}\, dr^2 + r^2 d\Omega^2_{2+n} \ , \qquad \phi(r) = 1 - \left(r_{S(4+n)}/r\right)^{1+n} ,$$
$$r_{S(4+n)} = \left[B_{(4+n)} G_{(4+n)} m\right]^{1/(1+n)} \ , \quad B_{(4+n)} = \tfrac{16\pi}{(2+n)A_{(2+n)}}, \quad A_{(2+n)} = \tfrac{2\pi^{(n+3)/2}}{\Gamma[(n+3)/2]}, \tag{1}$$

where $r_{S(4+n)}$ and $G_{(4+n)}$ are Schwarzschild radius, associated with the mass m, and gravitational constant in $(4+n)$-dimensional spacetime respectively. $A_{(2+n)}$ denotes the area of the unit $(2+n)$-sphere. The $(4+n)$-dimensional Planck length, time and mass are defined as follows

$$l_{Pl(4+n)} = t_{Pl(4+n)} = m^{-1}_{Pl(4+n)} = G^{1/(2+n)}_{(4+n)} \ . \tag{2}$$

Quantum fluctuations of spacetime , clocks, black holes and limits on computation in $(4+n)$*-dimensional spacetime.* – In this section, following the arguments proposed by Ng and van Dam [1], we analyze the thought experiment of distance and time measurement in the case of $(4+n)$ dimensions, write down explicit results for fluctuations of spacetime and consider implications for clocks, black holes and computation.

It is well known in general relativity that coordinates do not have any intrinsic meaning independent of observations; any coordinate system is defined only by explicitly carrying out spacetime distance measurements. In order to measure the distance between two points A and B, one puts a clock at A and a mirror at B. By sending a light signal from the clock to the mirror in timing experiment, one can determine the distance l. Let m be the mass of the clock. If the clock has the initial linear position spread δl_{in} when the light signal leaves it, then its final position spread grows to

$$\delta l = \delta l_{in} + l(m\delta l_{in})^{-1} \ , \tag{3}$$

when the light signal returns to the clock. The minimization of the final spread with respect to δl_{in} gives

$$\delta l \sim 2(l/m)^{1/2}, \tag{4}$$

so quantum mechanics alone suggests a massive clock to reduce the uncertainty in distance.

Now suppose in the measuring of distance one uses the light-clock consisting of a spherical cavity of diameter a, surrounded by a mirror wall of mass m, between which bounces a beam of light. The inevitable uncertainty in distance measurement caused by the clock is

$$\delta l \sim a = \beta r_{S(4+n)}. \tag{5}$$

Obviously, in order that the clock is not a black hole, one must suppose that the dimensionless quantity $\beta > 1$. The uncertainty (5) according to (1) increases with mass of clock, so the general relativity suggests a lightweight clock to do the measurement.

The total uncertainty in distance measurement can be written as the sum of (4) and (5)

$$\delta l_{tot} \sim 2(l/m)^{1/2} + \beta r_{S(4+n)}. \tag{6}$$

The minimization of (6) with respect to clock's mass gives the minimum uncertainty in distance

$$\delta l_{min} \sim C_{(4+n)} \beta^{(1+n)/(3+n)} \left(l l^{2+n}_{Pl(4+n)}\right)^{1/(3+n)} \ , \quad C_{(4+n)} = \left(\frac{(3+n)^{3+n}}{(1+n)^{1+n}} B_{(4+n)}\right)^{1/(3+n)} . \tag{7}$$

Taking the size of the clock as small as possible (i.e. $\beta \sim 1$) and neglecting numerical factors of order 1, for the uncertainty in distance measurement one gets

$$\delta l \sim \left(l l^{2+n}_{Pl(4+n)}\right)^{1/(3+n)} . \tag{8}$$

From eq.(8) one can also estimate the minimum measurable space distance by equating $\delta l = l$, and the minimum measurable length is $l_{\min} \sim l_{Pl(4+n)}$.

Considering similar gedanken experiment of measuring a time interval t and neglecting numerical factors, one easily gets an analogous expression for the corresponding uncertainty δt in time:

$$\delta t \sim \left(t t_{Pl(4+n)}^{2+n} \right)^{1/(3+n)} . \tag{9}$$

The uncertainties in distance and time can be translated into a metric uncertainties over a distance l and a time interval t caused by the quantum fluctuations in the fabric of spacetime

$$\delta g_{\mu\nu} \sim (l_{Pl(4+n)}/l)^{(2+n)/(3+n)}, \quad (t_{Pl(4+n)}/t)^{(2+n)/(3+n)} . \tag{10}$$

Using these results one can find upper bound on the lifetime of a simple clock [1]. Indeed, if the time resolution of the clock (i.e., the smallest time interval it is capable to measure) is τ, then its lifetime can be defined as the maximum measurable time interval T during which its time resolution exceeds the corresponding time fluctuation (9). Thus

$$\tau \geq \left(T t_{Pl(4+n)}^{2+n} \right)^{1/(3+n)}, \tag{11}$$

from which for the lifetime of the clock one gets

$$T \leq \tau \left(\tau / t_{Pl(4+n)} \right)^{2+n} . \tag{12}$$

If one considers a black hole as an ultimate clock, then its time resolution is given by the light travel time across the black hole's horizon, i.e. $\tau_{BH} \sim r_{S(4+n)}$, and from (12) the lifetime of the black hole is given by

$$T_{BH} \sim m_{Pl(4+n)}^{-1} (m/m_{Pl(4+n)})^{(3+n)/(1+n)}, \tag{13}$$

which is in accordance with the result previously obtained in [5].

The relation eq.(12) also can be used to put a limit on the memory space of any simple information processor. Indeed, the maximum number I of steps of information processing can be estimated by $I = T/\tau$, where T is the lifetime of processor with processing frequency $\nu = \tau^{-1}$, and so from eq.(12) one immediately gets

$$I \nu^{2+n} \leq \left(t_{Pl(4+n)}^{2+n} \right)^{-1}, \tag{14}$$

the bound which is universal in that it is independent of the mass, size and details of the simple computer.

And finally in this section let us investigate the holographic properties of spasetime fluctuations in $(4+n)$-dimensional spacetime. Consider a spatial region of volume $V_{(4+n)}$ ('hypercube' measuring $l \times l \times l \times \cdots \times l = l^{3+n}$). The number of degrees of freedom $N_{V_{(4+n)}}$ contained in the region is bound by the maximum number of the small hypercubes that can be put inside the region. But each side of the small hypercubes cannot be smaller than the accuracy δl with which can be measured each side l of taken spatial region, i.e. the side of small hypercube obeys eq.(8). Thus one has

$$N_{V_{(4+n)}} \sim l^{3+n} / \left(\delta l \right)^{3+n} \leq \left(l / l_{Pl(4+n)} \right)^{2+n}, \tag{15}$$

and so the uncertainties in distance caused by the quantum fluctuations satisfy the holographic counting of degrees of freedom.

Quantum fluctuations of spacetime and their implications in the ADD model. – In this section, by using the results introduced above, we consider quantum fluctuations of spacetime and their implications in the Arkani-Hamed-Dimopoulos-Dvali (ADD) model [2] with n extra spacelike compact dimensions of size L and low fundamental scale $m_{Pl((4+n))} \sim$ Tev. Within the framework of this model the links between 4- and $(4+n)$-dimensional physical values can be written as follows

$$G_{(4)} = \frac{G_{(4+n)}}{L^n} \;, \quad r_{S(4)} = \frac{B_{(4)}}{B_{(4+n)}} \frac{r^{1+n}_{S(4+n)}}{L^n}, \quad \frac{m^2_{Pl(4)}}{L^n} = m^{2+n}_{Pl(4+n)} \;, \; l^2_{Pl(4)} = \frac{l^{2+n}_{Pl(4+n)}}{L^n}. \quad (16)$$

It is reasonable to assume that $L \gg l_{Pl(4+n)}$, otherwise the extra dimensions would not have a classical spacetime structure. In this case from eq.(16) one gets $l_{Pl(4+n)} \gg l_{Pl(4)}$, and thus $L \gg l_{Pl(4+n)} \gg l_{Pl(4)}$.

Neglecting numerical factors of order 1 and using (16), from (8) one can estimate the distances at which fluctuations become comparable with the size L of compact extra dimensions, i.e. $\delta l \sim L$,

$$l \sim L(L/l_{Pl(4+n)})^{2+n} \quad \Rightarrow \quad l \sim L(L/l_{Pl(4)})^2. \quad (17)$$

Putting $m_{Pl(4+n)} = 1$ TeV and using $l_{Pl(4)} \sim 10^{-35}$ m for the size of extra dimensions one gets

$$L \sim 10^{30/n-19} \text{ m}, \quad (18)$$

and then, from (17) easily can be found

$$l \sim 10^{90/n+13} \text{ m} . \quad (19)$$

For $n = 2$ one gets $l \sim 10^{58}$ m, which is much greater then the size of the observable universe $l_{\text{universe}} \sim 10^{10}$ light-years $\sim 10^{26}$ m, thus in 6-dimensional spacetime fluctuations on any distances in the observable universe is much less as compared with the size of compact extra dimensions. We notice that for $n = 7$ one gets $l \sim l_{\text{Universe}}$, i.e. in case of 11-dimensional spacetime the fluctuations on the distances of the order of the size of observable universe become comparable with the size of compact dimensions.

One can readily estimate the size of clock to be used in the measurement of fluctuations $a/L \sim r_{S(4+n)}/L \sim (ll^2_{Pl(4)}/L^3)^{1/(3+n)}$, from which, using (18), even for the distances equal to the size of the observable universe in cases $n = 1$ and $n = 2$, one has $a/L \sim 10^{-19}$ and $a/L \sim 10^{-6}$ respectively. So in this cases $a \sim r_{S(4+n)} \ll L$. One can also estimate the mass of the clock $m \sim m_{Pl(4+n)}(l/l_{Pl(4+n)})^{(1+n)/(3+n)}$. In case $l \gg l_{Pl(4+n)}$ the clock's mass satisfies the inequality $m \gg m_{Pl(4+n)}$, i.e. the clock is well described in the framework of unquantized gravity.

In order to discuss holographic properties of fluctuations in distance in ADD model, one considers space region in the form of hypercube of characteristic size l. If $l < L$, then volume of the region $V_{(4+n)} \sim l^{3+n}$. Using (8) and neglecting numerical factors, for the number of degrees of freedom $N_{V_{(4+n)}}$ contained in the region one gets

$$N_{V_{(4+n)}} \sim l^{3+n} / \left(\delta l\right)^{3+n} \leq \left(l/l_{Pl(4+n)}\right)^{2+n}, \quad (20)$$

i.e. the uncertainties in distance satisfy the holographic principle in purely $(4+n)$-dimensional form. If the characteristic size of the hypercube $l > L$, then its volume $V_{(4+n)}$ (the region measures $l \times l \times l$ on the brane) is equal to $V_{(4+n)} \sim V_{(4)} V_{EtraSpase} \sim l^3 L^n$ (in the definition

of extra space volume $V_{ExtraSpace} \sim L^n$ the numerical factor of order 1, depending on the exact form of compact extra dimensions, is ignored). Thus, neglecting numerical factors, from eq.(8) for the number of bits in this region one gets

$$N_{V_{(4+n)}} \leq V_{(4+n)}/\delta l_{min}^{3+n} \sim l^3 L^n/(l l_{Pl(4+n)}^{2+n}) \sim l^3 L^n/(L^n l l_{Pl(4)}^2) \sim (l^2/l_{Pl(4)}^2), \tag{21}$$

i.e. the holographic principle is satisfied and effectively it has 4-dimensional form. Our conclusions about the holography in ADD model are different from those obtained in [3], where a different bound from gravity was used. ([1]) The origin of the difference we have considered in more details in our previous letter [6].

In conclusion, let us numerically estimate the effects of spacetime fluctuation in ADD model with n=2. In the first instance, in the case $n = 2$ and $L \sim 100\ \mu m$ on the size of the whole observable universe ($\sim 10^{10}$ light-years) the fluctuation in the ADD model is $\delta l_{(6)} \sim 10^{-11}$ m, whereas in the purely 4-dimensional case the fluctuation is $l_{(4)} \sim 10^{-15}$ m [1]. Furthermore, according to the classification proposed in [1], the ADD spacetime foam corresponds to the model with $\alpha = (2+n)/(3+n)$. Thus, following [1], one can also estimate the phase incoherence of light from extra-galactic source in this model

$$\Delta\phi_{(4+n)} \sim 2\pi \left(l_{Pl(4+n)}/\lambda\right)^{\alpha} (l/\lambda)^{1-\alpha}, \tag{22}$$

where λ is the wavelength of received radiation from a celestial source located at a distance l away. Using (18) it is straightforward to write

$$\Delta\phi_{(4+n)} \sim 10^{16-18/(3+n)} \left(l_{Pl(4)}/l\right)^{n/(9+3n)} \Delta\phi_{(4)}, \tag{23}$$

where $\Delta\phi_{(4)}$ is the phase incoherence in purely 4-dimensional spacetime [1]. For the active galaxy PKS1413+135, for which $\lambda \approx 1.6 \times 10^{-6}$ m and $l \approx 1.216$ Gpc $\approx 3 \times 10^{25}$ m, in case $n = 2$ one gets $\Delta\phi_{(6)} \sim 10^4 \Delta\phi_{(4)}$, i.e. the phase incoherence of light is greater by 4 orders in ADD model as compared with the purely 4-dimensional case. So it is worthwhile to estimate other possible effects of spacetime fluctuations in ADD model in the view of testing in experiments.

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([1])according to [3] holography is destroyed when extra dimensions are admitted; note that in [1] (last reference in it) Ng Y.J. also noticed the conclusions of [3] as different from his own results.